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1994 J. Phys. A: Math. Gen. 27 1791

(http://iopscience.iop.org/0305-4470/27/6/008)

J. Phys. A: Math. Gen. 27 (1994) 1791–1810. Printed in the UK

A random walk approach to spiral motion

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Received 2 September 1993, in final form 20 December 1993

Abstract. The paper deals with random spiral motion described in polar coordinates: the trajectory of a particle is described by the distance r from a given point as a function of the angle θ relative to a given direction. A stochastic description of the spiral shapes generated by the random variations of r and θ is suggested, based on the following assumptions: (*a*) the ray r is made up of additive contributions corresponding to certain jump processes; (*b*) the angle φ between two jumps is a random variable selected from a given probability law $g(\varphi)\,d\varphi$ with finite or infinite moments; (*c*) the contributions $\rho_1, \rho_2, \ldots$ of the different jumps to the ray r are independent random variables selected from a probability law $p(\rho)\,d\rho$ with finite moments. An expression for the generating functional of a random spiral $r(\theta)$ is given in the form of an infinite series which can be used to evaluate the stochastic properties of the ray r. The asymptotic properties of the random spiral shapes depend on the function $g(\varphi)\,d\varphi$: if all moments of the angle between two jumps exist and are finite, the average shape is a linear Archimedean spiral $\langle r(\theta)\rangle \sim \theta$ as $\theta \to \infty$, and the dispersion of the ray increases linearly with the angle $\langle \Delta r^2(\theta)\rangle \sim \theta$ as $\theta \to \infty$. If $g(\varphi)$ has a long tail $g(\varphi) \sim \varphi^{-(1+H)}$ as $\varphi \to \infty$ with $1 > H > 0$, the average shape is a nonlinear (fractal) Archimedean spiral $\langle r(\theta)\rangle \sim \theta^H$ as $\theta \to \infty$, and the fluctuations of the ray have an intermittent behaviour $\langle \Delta r^2(\theta)\rangle \sim \theta^{2H}$ as $\theta \to \infty$. A complete analysis is possible in a Markovian-like case for which the angle between two jumps is exponentially distributed. In this case a closed expression for the generating functional is available and all cumulants of the ray can be computed exactly: the mth cumulant which expresses the correlations among the rays at different angles $\theta_1, \ldots, \theta_m$ is proportional to the minimum angle $\min(\theta_1, \ldots, \theta_m)$ and the intermittent behaviour is missing. An alternative stochastic description is suggested based on the assumption that the number of jumps in a given angle interval is distributed according to an inhomogeneous Poisson law. This model is also analytically tractable; it is less restrictive in the sense that the average shape corresponds to a broader class of spirals including the linear and the fractal Archimedean and the logarithmic ones; however, it cannot be used to describe the intermittent behaviour.

1. Introduction

Spiral motions and shapes have long been studied in specialized areas of science and technology ranging from mechanics, astronomy and astrophysics to chemistry, biochemistry and anatomy (Seiden and Schulman 1990, Saslaw 1985, Murray 1989, Wesfreid *et al* 1988 and references therein). Quite recently a more general interest has appeared with new approaches of potentially wider applicability being proposed (Davis 1993 and references therein). The spirals are intimately connected with various aspects

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of nonlinear dynamics from nonlinear wave propagation (Wesfried *et al* 1988) to the geometrical and statistical fractals (West 1990). In spite of their ubiquity, the theoretical descriptions of the spiral motions and shapes corresponding to different processes are based on different approaches. As far as we know in this field, a unitary mathematical description is still missing.

The aim of this paper is to outline the possibilities of the stochastic description of spiral shapes or motions by means of the formalism of random walks. Although random walks are commonly used for describing various natural processes (Weiss and Rubin 1983, Haus and Kehr 1987, Bouchaud and Georges 1990 and references therein), their possible applicability to the study of spiral motions or shapes has been ignored. The underlying idea used in this paper is to describe a spiral as a random walk in polar variables: ray length–polar angle. The plan of the paper is as follows. In section 2 we present some basic notions related to the description of a spiral in terms of a random walk. In section 3 we discuss a simple model for which a complete analytical solution is available. In section 4 a renewal-type generalization of the model introduced in section 3 is analysed. In section 5 we suggest another generalization based on the use of an inhomogeneous Poisson process. Finally, in section 6 a comparison among the approaches introduced in this paper is made and the possibilities of application are analysed.

2. Basic notions

A plane spiral is usually described in polar coordinates

$$r=r(\theta) \tag{1}$$

where r is the distance (the length of the ray r) from a given point (the centre) to a current point of the spiral and θ is the angle made by the ray r with a given direction. If function (1) is periodical with a period of 2π, then the corresponding shape is a closed contour. A spiral corresponds to the case where $r=r(\theta)$ monotonically increases with the increase in the angle θ.

We are not interested here in the 'deterministic' spirals corresponding to a given function $r(\theta)$ but rather in the stochastic spirals for which $r(\theta)$ is a random function; just as for a deterministic spiral, for a random spiral a given realization $r(\theta)$ should be a non-decreasing function of θ. The properties of a stochastic spiral may be described by the probability density functional

$$\phi[r(\theta)]D[r(\theta)] \tag{2a}$$

$$\overline{\underline{\iint}}\, \phi[r(\theta)]D[r(\theta)]=1 \tag{2b}$$

where $D[r(\theta)]$ is an integration measure over the space of functions $r(\theta)$ and $\overline{\underline{\iint}}$ stands for functional integration. The main property of interest of a stochastic spiral is the average value of the ray length corresponding to a given angle,

$$\langle r(\theta)\rangle=\overline{\underline{\iint}}\, r(\theta)\phi[r(\theta)]D[r(\theta)]. \tag{3}$$

The superior moments are also of interest: they are a measure of the shape fluctuations. We use the central moments

$$\langle r(\theta_1)r(\theta_2)\rangle = \overline{\iint} r(\theta_1)r(\theta_2)\phi[r(\theta)]D[r(\theta)]\ldots \tag{4}$$

and the cumulants

$$\langle\langle r(\theta_1)r(\theta_2)\rangle\rangle = \langle r(\theta_1)r(\theta_2)\rangle - \langle r(\theta_1)\rangle\langle r(\theta_2)\rangle\ldots \tag{5}$$

The computation of these moments is easier in terms of the generating functional

$$\begin{aligned} Z[\eta(\theta)] &= \overline{\iint} \exp\left(-\int \eta(\theta)r(\theta)\,\mathrm{d}\theta\right)\phi[r(\theta)]D[r(\theta)] \\ &= \left\langle \exp\left(-\int \eta(\theta)r(\theta)\,\mathrm{d}\theta\right)\right\rangle \end{aligned} \tag{6}$$

where $\eta(\theta)$ is a suitable test function of θ. In terms of $Z[\eta(\theta)]$ the central moments and the cumulants are given by the functional derivatives

$$\langle r(\theta_1)\ldots r(\theta_m)\rangle = \left.\frac{\delta^m Z[\eta(\theta)]}{\delta\eta(\theta_1)\ldots\delta\eta(\theta_m)}\right|_{\eta(\theta)\equiv 0} \tag{7}$$

and

$$\langle\langle r(\theta_1)\ldots r(\theta_m)\rangle\rangle = \left.\frac{\delta^m \ln Z[\eta(\theta)]}{\delta\eta(\theta_1)\ldots\delta\eta(\theta_m)}\right|_{\eta(\theta)\equiv 0}. \tag{8}$$

By borrowing an idea from the random walk theory (Weiss and Rubin 1983, Haus and Kehr 1987) we assume that the random spirals are generated by the contributions of many independent increments $\rho_0, \rho_1, \ldots, \rho_m, \ldots$ selected from a certain probability law

$$p(\rho)\,\mathrm{d}\rho \tag{9a}$$

$$\int_0^\infty p(\rho)\,\mathrm{d}\rho = 1 \tag{9b}$$

with finite moments. A given realization is made up of a succession of jumps. For each jump an additional increment to the value of the ray length is added; after the occurrence of q jumps the ray length r_q is equal to

$$r_q = r_{q-1} + \rho_q = \rho_0 + \rho_1 + \ldots + \rho_q \tag{10}$$

where r_{q-1} is the ray length after $q-1$ jumps, ρ_0 is the initial value of the ray length and $\rho_1, \ldots, \rho_m$ are the increments corresponding to the different jumps. Between two jumps the ray length is constant. The jumps take place at different angles $\theta_1, \theta_2, \ldots, \theta_q, \ldots$. The different models presented in this paper correspond to different stochastic properties of the angles $\theta_1, \theta_2, \ldots$.

3. Markovian spiral shapes

We start out by considering the Markovian spiral shapes for which the angles $\varphi_1, \varphi_2, \ldots, \varphi_q, \ldots$ between two successive jumps are independent random variables selected from an exponential probability law

$$g(\varphi)\,\mathrm{d}\varphi \tag{11a}$$

$$\int_0^\infty g(\varphi)\,\mathrm{d}\varphi = 1 \tag{11b}$$

$$g(\varphi) \sim \exp[-\mathrm{const}\;\varphi]. \tag{11c}$$

The absolute angles $\theta_1, \theta_2, \ldots$ at which the jumps occur may be expressed in terms of the relative angles $\varphi_1, \varphi_2, \ldots$ by means of a recursive relationship similar to equation (10):

$$\theta_q = \theta_{q-1} + \varphi_q = \varphi_0 + \varphi_1 + \ldots + \varphi_q. \tag{12}$$

We notice the similarity with a continuous time-directed Markovian random walk of a particle in one-dimensional space; the ray length corresponds to the position of the moving particle and the angle to the time. However, this analogy is rather formal and quite limited. In the random walk theory we are interested only in the moments of the position of the particle at a given time. On the other hand, the stochastic properties of a spiral shape are described in terms of the probability density functional $\phi[r(\theta)]D[r(\theta)]$. In the random walk theory little attention is paid to the probadensity functional of the position of the particle as a function of time. Most approaches do not even use such a notion.

The probability density functional can be expressed as an average of a delta functional symbol

$$\delta\left[r(\theta) - \sum_{l=0}^{q} \rho_l\right] D[r(\theta)] \tag{13}$$

which corresponds to the superposition law (10) over all possible values of the increments $\rho_0, \rho_1, \ldots, \rho_q, \ldots$ and over the angles of occurrence of the different jumps $\theta_1, \ldots, \theta_q, \ldots$ and over all possible numbers of jumps. In order to evaluate this average, for a given succession of angles $\theta_q \geqslant \theta_{q-1} \geqslant \ldots \geqslant \theta_1$ we introduce the probability

$$\psi_q(\theta_1, \ldots, \theta_q)\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q \tag{14a}$$

$$\sum_{q=0}^{\infty} \int_0^{\theta} \int_0^{\theta_q} \ldots \int_0^{\theta_2} \psi_q(\theta_1, \ldots, \theta_q)\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q = 1 \tag{14b}$$

that there are q jumps and that the angles of occurrence of these jumps are between θ_1 and $\theta_1 + \mathrm{d}\theta_1, \ldots$ and θ_q and $\theta_q + \mathrm{d}\theta_q$, respectively. This probability density can be evaluated in terms of the probability density $g(\varphi)\,\mathrm{d}\varphi$ of the angle between two successive jumps. As the angles $\varphi_1, \ldots, \varphi_q$ are independent random variables, we have

$$\psi_q(\theta_1, \ldots, \theta_q)\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q = g(\theta_1) g(\theta_2 - \theta_1) \ldots g(\theta_q - \theta_{q-1}) f(\theta - \theta_q)\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q \tag{15}$$

where

$$f(\varphi)=\int_{\varphi}^{\infty} g(\varphi)\,\mathrm{d}\varphi \tag{16}$$

is the probability that no jumps occur in the angle interval 0, φ.

By using the function ψ_q the average of the functional delta symbol (13) can be evaluated easily:

$$\begin{aligned}\phi[r(\theta)]D[r(\theta)]\\ &=\sum_{q=0}^{\infty}\int_{\rho_0}\cdots\int_{\rho_q}\int_0^{\theta}\cdots\int_0^{\theta_3}\int_0^{\theta_2}\delta\left[r(\theta)-\sum_{l=0}^{q}\rho_l\right]D[r(\theta)]\\ &\quad\times\psi_q(\theta_1,\ldots,\theta_q)p(\rho_0)p(\rho_1)\ldots p(\rho_q)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_q\,\mathrm{d}\rho_0\ldots\mathrm{d}\rho_q.\end{aligned} \tag{17}$$

By combining equations (6), (15) and (17) we get the following expression for the generating functional of $\phi[r(\theta)]D[r(\theta)]$:

$$\begin{aligned}Z[\eta(\theta)]&=f(\theta)\bar{p}\left(\int_0^{\theta}\eta(\theta)\,\mathrm{d}\theta\right)\\ &\quad+\sum_{q=1}^{\infty}\int_0^{\theta}\cdots\int_0^{\theta_2}g(\theta_1)g(\theta_1-\theta_2)\ldots g(\theta_q-\theta_{q-1})f(\theta-\theta_q)\\ &\quad\times\bar{p}\left(\int_0^{\theta}\eta(\theta)\,\mathrm{d}\theta\right)\bar{p}\left(\int_{\theta_1}^{\theta}\eta(\theta)\,\mathrm{d}\theta\right)\ldots\bar{p}\left(\int_{\theta_q}^{\theta}\eta(\theta)\,\mathrm{d}\theta\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_q\end{aligned} \tag{18}$$

where

$$\bar{p}(s)=\int_0^{\infty}p(\rho)\exp(-s\rho)\,\mathrm{d}\rho \tag{19}$$

is the Laplace transform of the probability distribution $p(\rho)$. The details of the derivation of equation (18) are presented in appendix 1.

Equation (18) is valid for an arbitrary probability density $g(\varphi)$. If $g(\varphi)$ is given by the exponential distribution (11*c*), then equation (18) is simplified. The first step is to express equation (11*c*) in the standard form:

$$g(\varphi)=\langle\varphi\rangle^{-1}\exp(-\varphi/\langle\varphi\rangle) \tag{20}$$

where

$$\langle\varphi\rangle=\int_0^{\infty}\varphi g(\varphi)\,\mathrm{d}\varphi \tag{21}$$

is the average angle between two jumps. The probability $f(\varphi)$ can be easily computed,

$$f(\varphi)=\int_{\varphi}^{\infty}g(\varphi)\,\mathrm{d}\varphi=\exp(-\varphi/\langle\varphi\rangle). \tag{22}$$

The next step is to change the integration limits over $\theta_1, \ldots, \theta_q$ by assuming that there are no restrictions concerning the relative positions of the angles at which the jumps occur. For that we use the relation

$$\int_0^{\theta} \ldots \int_0^{\theta_2} \ldots \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q = \frac{1}{q!} \int_0^{\theta} \ldots \int_0^{\theta} \ldots \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q \tag{23}$$

which is commonly used in statistical physics. By removing the restrictions $\theta_q \geqslant \theta_{q-1} \geqslant \ldots \geqslant \theta_1$ the positions of the different jumps are equivalent and a $1/q!$ Gibbs factor should be introduced in the RHS of equation (23). For a proof of equation (23), see Chandrasekhar and Münch (1950). By inserting the expressions (20) and (22) for $g(\varphi)$ and $f(\varphi)$ into equation (18) and using equation (23), we arrive at

$$\begin{aligned} Z[\eta(\theta)] = {} & \bar{p}\left(\int_0^{\theta} \eta(\theta)\,\mathrm{d}\theta\right) \exp[-\nu(\theta)] + \bar{p}\left(\int_0^{\theta} \eta(\theta)\,\mathrm{d}\theta\right) \exp[-\nu(\theta)] \\ & \times \sum_{q=1}^{\infty} \frac{1}{q!} [\nu(\theta)]^q \int_0^{\theta} \ldots \int_0^{\theta} \frac{\mathrm{d}\theta_1}{\theta} \ldots \frac{\mathrm{d}\theta_q}{\theta} \prod_{l=1}^{q} \bar{p}\left(\int_{\theta_l}^{\theta} \eta(\theta)\,\mathrm{d}\theta\right) \end{aligned} \tag{24}$$

where

$$\nu(\theta) = \theta / \langle \varphi \rangle. \tag{25}$$

By examining equation (24) we see that it is in fact the expansion of an exponential,

$$Z[\eta(\theta)] = \bar{p}\left(\int_0^{\theta} \eta(\theta)\,\mathrm{d}\theta\right) \exp\left\{-\nu(\theta) + \frac{\nu(\theta)}{\theta} \int_0^{\theta} \mathrm{d}\varphi\, \bar{p}\left(\int_{\varphi}^{\theta} \eta(\theta)\,\mathrm{d}\theta\right)\right\}. \tag{26}$$

In order to clarify the significance of the parameter $\nu(\theta)$ we compute the function $\psi_q(\theta_1, \ldots, \theta_q)$ by using equations (20) and (22). We get

$$\psi_q(\theta_1, \ldots, \theta_q)\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q = [\nu(\theta)]^q \exp[-\nu(\theta)] \frac{\mathrm{d}\theta_1}{\theta} \ldots \frac{\mathrm{d}\theta_q}{\theta}. \tag{27}$$

By integrating this equation over all possible values of $\theta_1, \ldots, \theta_q$ we obtain the probability $R(q)$ that there are q jumps in an angle interval of length θ:

$$R(q) = [q!]^{-1} [\nu(\theta)]^q \exp[-\nu(\theta)]. \tag{28}$$

$R(q)$ is a Poissonian with a parameter $\nu(\theta)$. Thus, $\nu(\theta)$ is the mean number of jumps in an angle interval of length θ.

Expression (26) for the generating functional $Z[\eta(\theta)]$ contains all the information necessary for describing the stochastic properties of the random shape $r(\theta)$. In order to extract this information we use the following expansions of the Laplace transform of the probability density $p(\rho)$:

$$\bar{p}(s) = \sum_{m=0}^{\infty} (-1)^m \frac{\langle \rho^m \rangle}{m!} s^m \tag{29}$$

and

$$\bar{p}(s) = \exp\left\{\sum_{m=1}^{\infty} (-1)^m \frac{\langle\langle \rho^m \rangle\rangle}{m!} s^m\right\} \tag{30}$$

where $\langle\rho^m\rangle$ and $\langle\!\langle\rho^m\rangle\!\rangle$ are the central moments and the cumulants of the increment of the ray length corresponding to a step, respectively. By inserting equations (29) and (30) into equation (26) we obtain

$$Z[\eta(\theta)]=\exp\left\{\sum_{m=1}^{\infty}(-1)^m\frac{\langle\!\langle\rho^m\rangle\!\rangle}{m!}\int_0^\theta\dots\int_0^\theta\eta(\theta_1)\dots\eta(\theta_m)\,\mathrm{d}\theta_1\dots\mathrm{d}\theta_m+\frac{\nu(\theta)}{\theta}\right.$$
$$\left.\times\sum_{m=1}^{\infty}(-1)^m\frac{\langle\rho^m\rangle}{m!}\int_0^\theta\mathrm{d}\varphi\int_\varphi^\theta\dots\int_\varphi^\theta\eta(\theta_1)\dots\eta(\theta_m)\,\mathrm{d}\theta_1\dots\mathrm{d}\theta_m\right\}. \quad (31)$$

In appendix 2 we prove the relationship

$$\begin{aligned}I_m&=\int_0^\theta\mathrm{d}\varphi\int_\varphi^\theta\mathrm{d}\theta_1\dots\int_\varphi^\theta\mathrm{d}\theta_m\dots\\&=\int_0^\theta\mathrm{d}\theta_1\dots\int_0^\theta\mathrm{d}\theta_m\int_0^{\theta_m^*}\mathrm{d}\varphi\dots\end{aligned}\quad (32)$$

where

$$\theta_m^*=\min(\theta_1,\dots,\theta_m)\quad (33)$$

is the minimum angle from the set $(\theta_1,\dots,\theta_m)$. By using equation (32) we can rewrite equation (33) in the standard form of a cumulant expansion,

$$Z[\eta(\theta)]=\exp\left\{\sum_{m=1}^{\infty}(-1)^m\frac{1}{m!}\int_0^\theta\dots\int_0^\theta\eta(\theta_1)\dots\eta(\theta_m)\right.$$
$$\left.\times\left[\langle\!\langle\rho^m\rangle\!\rangle+\frac{1}{\langle\varphi\rangle}\langle\rho^m\rangle\min(\theta_1,\dots,\theta_m)\right]\mathrm{d}\theta_1\dots\mathrm{d}\theta_m\right\}.\quad (34)$$

By evaluating from equation (34) the functional derivatives (8) we get the following expressions for the cumulants of the ray length:

$$\langle\!\langle r(\theta_1)\dots r(\theta_m)\rangle\!\rangle=\langle\!\langle\rho^m\rangle\!\rangle+\frac{1}{\langle\varphi\rangle}\langle\rho^m\rangle\min(\theta_1,\dots,\theta_m).\quad (35)$$

In particular, the average value of the ray length at an angle θ and the correlation function of the ray lengths at the angles θ_1 and θ_2 are given by

$$\langle r(\theta)\rangle=\langle\rho\rangle[1+\theta/\langle\varphi\rangle]\quad (36)$$

$$\begin{aligned}\langle\Delta r(\theta_1)\Delta r(\theta_2)\rangle&=\langle\!\langle r(\theta_1)r(\theta_2)\rangle\!\rangle\\&=\langle\Delta\rho^2\rangle+[\langle\Delta\rho^2\rangle+\langle\rho\rangle^2]\frac{\min(\theta_1,\theta_2)}{\langle\varphi\rangle}.\end{aligned}\quad (37)$$

We note that the asymptotic behaviour of the average shape for large angles is given by an Archimedean spiral:

$$\langle r(\theta)\rangle\simeq[\langle\rho\rangle/\langle\varphi\rangle]\theta\qquad\text{as }\theta\to\infty\quad (38)$$

Similarly, the dispersion of the particle shape increases linearly with the increase in the angle,

$$\langle \Delta r^2(\theta) \rangle \simeq [\langle \rho^2 \rangle / \langle \varphi \rangle]\theta \qquad \text{as } \theta \to \infty \tag{39}$$

and the relative fluctuation $[\langle \Delta r^2(\theta) \rangle]^{1/2} / \langle r(\theta) \rangle$ tends to 0 as $\theta^{-1/2}$ as $\theta \to \infty$,

$$[\langle \Delta r^2(\theta) \rangle]^{1/2} / \langle r(\theta) \rangle = [\langle \varphi \rangle^{1/2} \langle \rho^2 \rangle^{1/2} / \langle \rho \rangle]\theta^{-1/2} \qquad \text{as } \theta \to \infty. \tag{40}$$

Thus the relative fluctuations of the particle shape become negligible as $\theta \to \infty$.

The joint probability density

$$P_q(r_q, \theta_q; \ldots; r_1, \theta_1)\, \mathrm{d}r_q \ldots \mathrm{d}r_1 \tag{41a}$$

$$\int \ldots \int P_q\, \mathrm{d}r_q \ldots \mathrm{d}r_1 = 1 \tag{41b}$$

that at the angle θ_1 the ray length has a value between r_1 and $r_1 + \mathrm{d}r_1, \ldots$, and that at the angle θ_q the ray length has a value between r_q and $r_q + \mathrm{d}r_q$, can also be evaluated from the generating functional. We introduce the multiple Laplace transform

$$\bar{P}_q(s_q, \theta_q; \ldots; s_1, \theta_1) = \int_0^\infty \ldots \int \exp\left(-\sum s_q r_q\right) P_q\, \mathrm{d}r_q \ldots \mathrm{d}r_1. \tag{42}$$

From the definition (6) of the generating functional we note that we have

$$\bar{P}_q(s_q, \theta_q; \ldots; s_1, \theta_1) = Z[\eta(\theta) = \sum s_l \delta(\theta - \theta_l)]. \tag{43}$$

Thus, at least in principle, P_q may be evaluated from equations (26) and (43) by means of a multiple inverse Laplace transformation. In particular, we get the following expression for the Laplace transform $\bar{P}_1(s, \theta)$ of the one-point probability density of the ray length:

$$\bar{P}_1(s, \theta) = \bar{p}(s) \exp[(\bar{p}(s) - 1)/\langle \varphi \rangle]. \tag{44}$$

Relationship (44) may be obtained by applying the Laplace transform to the Markovian master equation

$$\langle \varphi \rangle \partial_\theta P_1(r, \theta) = \int_0^r p(\rho) P_1(r - \rho, \theta)\, \mathrm{d}\rho - P_1(r, \theta) \tag{45}$$

with the initial condition

$$P_1(r, \theta = 0) = p(r) \tag{46}$$

which results by using the analogy to the continuous-time random walk.

4. Non-Markovian spiral shapes

A non-Markovian stochastic spiral results by replacing the exponential distribution $g(\varphi)$ (equation (11*b*) or (20)) by an arbitrary probability density having either finite or infinite moments. In this case, equations (15)–(18) and (41)–(43) derived in the preceding section remain valid. A closed expression for the generating functional $Z[\eta(\theta)]$ is no longer available. However, the central moments may be derived step by step from equation (18) by evaluating the functional derivatives (7) and applying the Laplace transform to the results. Inserting equation (18) into equation (7), after some standard calculations, we obtain

$$\begin{aligned}\mathscr{L}\langle r(\theta)\rangle &= \int_0^\infty \langle r(\theta)\rangle \exp(-x\theta)\,\mathrm{d}\theta \\ &= \sum_{q=0}^{\infty} (q+1)\bar{p}^q(s)\left(\int_0^\infty \exp(-\rho s)\rho p(\rho)\,\mathrm{d}\rho\right)x^{-1}\,\bar{g}^q(x)[1-\bar{g}(x)]|_{s=0} \\ &= \langle\rho\rangle/[x(1-\bar{g}(x)]\end{aligned} \tag{47}$$

$$\begin{aligned}\mathscr{L}\langle r^2(\theta)\rangle &= \sum_{q=1}^{\infty} (q+1)q\bar{p}^{q-1}(s) \\ &\quad\times\left(\int_0^\infty \exp(-\rho s)\rho p(\rho)\,\mathrm{d}\rho\right)^2 x^{-1}\bar{g}^q(x)[1-\bar{g}(x)]|_{s=0} \\ &\quad+\sum_{q=0}^{\infty} (q+1)\bar{p}^q(s)\left(\int_0^\infty \exp(-\rho s)\rho^2 p(\rho)\,\mathrm{d}\rho\right)x^{-1}\bar{g}^q(x)[1-g(x)]|_{s=0} \\ &= \langle\rho^2\rangle/[x(1-\bar{g}(x))]+2\bar{g}(x)\langle\rho\rangle^2/[x(1-\bar{g}(x))^2].\end{aligned} \tag{48}$$

Here we have used the Laplace transform of equation (16),

$$\mathscr{L}f(\theta)=x^{-1}[1-\bar{g}(x)] \qquad \bar{g}(x)=\mathscr{L}g(\theta)=\int_0^\infty \mathrm{e}^{-x\theta}g(\theta)\,\mathrm{d}\theta \tag{49}$$

and the fact that after the evaluation of the functional derivatives in the resulting expressions the multiple integrals over $\theta_1, \ldots, \theta_q$ have the structure of a multiple convolution product:

$$\begin{aligned}&\int_0^\infty \exp(-x\theta)\int_0^\theta \ldots \int_0^{\theta_2} g(\theta_1)\ldots g(\theta_q-\theta_{q-1})f(\theta-\theta_q)\,\mathrm{d}\theta_1\ldots \mathrm{d}\theta_q\,\mathrm{d}\theta \\ &\qquad = \mathscr{L}[(g(\theta)\otimes)^q\otimes f(\theta)]=\bar{g}^q(x)[1-\bar{g}(x)]/x\end{aligned} \tag{50}$$

where $\otimes$ denotes the convolution product over angles.

The asymptotic behaviour of the first two moments of the shape depends on the form of the function $g(\varphi)$. By making an analogy to the continuous-time random walk theory (Shlesinger 1974) we distinguish the following two cases:

(*a*) all moments of $g(\varphi)$ exist and are finite. For $x\to 0$, which corresponds to $\varphi\to\infty$, we have

$$1-\bar{g}(x)=\langle\varphi\rangle x+\mathrm{o}(x^2) \qquad \text{as } x\to 0 \tag{51}$$

(*b*) $g(\varphi)$ has a long tail of the inverse power law form

$$g(\varphi)\sim\varphi^{-(H+1)} \qquad \text{as } \varphi\to\infty \text{ with } 1>H>0. \tag{52}$$

The latter situation corresponds to a statistical fractal characterized by the scaling exponent H. The Laplace transform $\bar{g}(x)$ has the following behaviour as $s\to 0$:

$$1-\bar{g}(x)\simeq Bx^H \qquad \text{as } x\to 0 \qquad \text{with} \qquad B>0. \tag{53}$$

By inserting equations (51) and (53) into equations (49) and (50) and coming back to the angle variable we can evaluate the asymptotic behaviour of the first two moments as $\theta\to\infty$. In case (*a*) we recover the relationships (38) and (39) derived in the preceding section; it follows that the average shape for $\theta\to\infty$ corresponds to a linear Archimedean spiral and the relative fluctuations of the shape become negligible as $\theta\to\infty$.

In case (*b*) the Laplace transforms of the first two moments may be approximated by

$$\begin{aligned}&\mathscr{L}\langle r(\theta)\rangle = B^{-1}x^{-(H+1)} \qquad \text{as } x\to 0\\ &\mathscr{L}\langle r^2(\theta)\rangle = \langle\rho^2\rangle B^{-1}x^{-(H+1)} + 2\langle\rho\rangle^2\bar{g}(x)B^{-2}x^{-(2H+1)} \qquad \text{as } x\to 0\end{aligned} \tag{54}$$

from which, by coming back to the angle variable and by keeping the dominant terms, we obtain

$$\langle r(\theta)\rangle \simeq \frac{\langle\rho\rangle\theta^H}{B\Gamma(H+1)} \qquad \text{as } \theta\to\infty \tag{55}$$

$$\langle\Delta r^2(\theta)\rangle = \langle r^2(\theta)\rangle - \langle r(\theta)\rangle^2 \simeq \frac{\langle\rho\rangle^2\theta^{2H}}{B^2\Gamma^2(H+1)}\,\varepsilon(H) \qquad \text{as } \theta\to\infty \tag{56}$$

where $\Gamma(y)$ is the complete gamma function and

$$\varepsilon(H) = \frac{\sqrt{\pi}\,\Gamma(H+1)}{2^{2H-1}\Gamma(H+\frac{1}{2})} - 1. \tag{57}$$

Here we have used the well-known duplication formula of the gamma function,

$$\Gamma(2H) = \pi^{-1/2}\,2^{2H-1}\Gamma(H)\Gamma(H+\tfrac{1}{2}). \tag{58}$$

The average shape for $\theta\to\infty$ corresponds to a nonlinear Archimedean spiral (equation (55)). The increase of the average ray with the angle θ is slower than in the Markovian case. The relative fluctuation

$$\langle\Delta r^2(\theta)\rangle^{1/2}/\langle r(\theta)\rangle \to [\varepsilon(H)]^{1/2} \qquad \text{as } \theta\to\infty \tag{59}$$

tends towards a constant value rather than decreasing to zero. This illustrates the intermittent behaviour of the fluctuations of the shape. The scaling exponent of the increase in the dispersion $\langle\Delta r^2(\theta)\rangle$ with the angle θ is two times bigger than the scaling exponent corresponding to the average value $\langle r(\theta)\rangle$. It follows that in this case the shape fluctuations are not negligible in the limit $\theta\to\infty$. These results are valid for any value of the exponent H between 0 and 1 $(1>H>0)$. In this range the function $\varepsilon(H)$ is always positive and thus equations (56) and (59) are physically consistent.

The joint probability densities $P_q(r_q, \theta_q; \ldots; r_1, \theta_1)$ (equation (41)) may be evaluated by applying equations (18), (42) and (43). As in the preceding section, we limit ourselves to the evaluation of the one-point probability density $P_1(r, \theta)$. By combining equations (18), (42) and (43) we get the following expression for the double Laplace transform of $P_1(r, \theta)$:

$$\begin{aligned}\bar{\bar{P}}_1(s, x) &= \int_0^\infty\int_0^\infty \exp(-rs-\theta x)P_1(r, \theta)\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \sum_{q=0}^{\infty} \bar{p}^{q+1}(s)x^{-1}\,\bar{g}^q(x)[1-\bar{g}(x)]\\ &= \bar{p}(s)[1-\bar{g}(x)]/[x(1-\bar{g}(x)\bar{p}(s))].\end{aligned} \tag{60}$$

This equation can also be obtained as the solution of a master equation of a non-Markovian type. We introduce the rate of occurrence of a jump in an angle interval φ, $\varphi+\mathrm{d}\varphi$:

$$\lambda(\varphi)\,\mathrm{d}\varphi=g(\varphi)\,\mathrm{d}\varphi/f(\varphi). \tag{61}$$

We define the probability

$$\mathscr{P}_1(r, \varphi, \theta)\,\mathrm{d}r\,\mathrm{d}\varphi \tag{62a}$$

$$\int_0^\infty\int_0^\theta \mathscr{P}_1(r, \varphi, \theta)\,\mathrm{d}r\,\mathrm{d}\varphi=1 \tag{62b}$$

that at the absolute angle θ the ray length is between r and $r+\mathrm{d}r$ and that the angle interval from the last jump is between φ and $\varphi+\mathrm{d}\varphi$. In terms of $\lambda(\varphi)\,\mathrm{d}\varphi$ we can write a set of balance equations for $\mathscr{P}_1(r, \varphi, \theta)$ which is similar to the system of age-dependent master equations from the random walk theory (ADME, Vlad and Pop 1989a, b):

$$(\partial_\theta+\partial_\varphi)\mathscr{P}_1(r, \varphi, \theta)=-\lambda(\varphi)\mathscr{P}_1(r, \varphi, \theta) \tag{63}$$

$$\mathscr{P}_1(r, \varphi=0, \theta)=\int_0^r\int_0^\theta \lambda(\varphi')p(\rho)\mathscr{P}_1(r-\rho, \varphi', \theta)\,\mathrm{d}\rho\,\mathrm{d}\varphi' \tag{64}$$

with the initial condition

$$\mathscr{P}_1(r, \varphi, \theta=0)=p(r)\delta(\varphi). \tag{65}$$

By integrating $\mathscr{P}_1(r, \varphi, \theta)$ over all values of φ we should get $P_1(r, \theta)$:

$$P_1(r, \theta)=\int_0^\theta \mathscr{P}_1(r, \varphi, \theta)\,\mathrm{d}\varphi. \tag{66}$$

In appendix 3 we show that by solving equations (63)–(66) for $P_1(r, \theta)$ we recover equation (60).

5. An alternative approach

At present the catalogue of possible average shapes is rather poor: we can either get a linear or a nonlinear Archimedean spiral. In order to enlarge the catalogue of possible spiral shapes we shall introduce a generalized model based on the use of inhomogeneous Poisson statistics. We assume that the probability $R(q)$ that there are q jumps in an angle interval of absolute length θ is given by a relationship similar to equation (28),

$$R(q\,|\,\theta)=[\nu(\theta)]^q(q!)^{-1}\exp[-\nu(\theta)] \tag{67}$$

where $\nu(\theta)$ is an arbitrary non-decreasing function of θ, not necessarily a linear one. The position θ' of any jump occurring in the angle integral of length θ is an independent random variable selected from a given probability law,

$$\xi(\theta'\,|\,\theta)\,\mathrm{d}\theta' \tag{68a}$$

$$\int_0^\theta \xi(\theta'\,|\,\theta)\,\mathrm{d}\theta'=1. \tag{68b}$$

Although the positions $\theta'_1, \ldots, \theta'_q$ of the jumps are independent of each other, they are all selected from the probability law (68*a*) which depends on the length θ of the angle interval considered. In the present description a new feature arises which is missing in previous approaches. The probability density $\xi(\theta' \mid \theta)$ depends both on the current angle θ' and on the total length θ of the angle interval considered, and the jumps are inhomogeneously distributed. In order to make a distinction between these two types of angles in this section, the current angles bear a prime (θ', $\theta'_1, \ldots, \theta'_q$, etc).

The probability density functional $\phi[r(\theta')]D[r(\theta')]$ of the shape may be evaluated as an average of the delta functional symbol (13),

$$\phi[r(\theta')]D[r(\theta')] = \sum_{q=0}^{\infty} \int_{\rho_0} \cdots \int_{\rho_q} \int_0^{\theta} \int_0^{\theta} \cdots \int_0^{\theta} \delta\left[r(\theta') - \sum_{l=0}^{q} \rho_l\right] D[r(\theta')]$$

$$\times [v(\theta)]^q (q!)^{-1} \exp[-v(\theta)] p(\rho_0)\, \mathrm{d}\rho_0 \ldots p(\rho_q)\, \mathrm{d}\rho_q$$

$$\times \xi(\theta'_1 \mid \theta)\, \mathrm{d}\theta'_1 \ldots \xi(\theta'_q \mid \theta)\, \mathrm{d}\theta'_q. \tag{69}$$

The corresponding generating functional can be evaluated in the same way as in section 3. We have

$$Z[\eta(\theta')] = \exp[-v(\theta)] \bar{p}\left(\int_0^{\theta} \eta(\theta')\, \mathrm{d}\theta'\right) \left\{1 + \sum_{q=1}^{\infty} [v(\theta)]^q (q!)^{-1}\right.$$

$$\left. \times \prod_{l=1}^{q} \left[\int_0^{\theta} \xi(\theta'_l \mid \theta) \bar{p}\left(\int_{\theta'_l}^{\theta} \eta(\theta' \mid \theta)\, \mathrm{d}\theta'\right) \mathrm{d}\theta'_l\right]\right\}$$

$$= \bar{p}\left(\int_0^{\theta} \eta(\theta')\, \mathrm{d}\theta'\right) \exp\left\{-v(\theta) + v(\theta) \int_0^{\theta} \xi(\varphi \mid \theta)\right.$$

$$\left. \times \bar{p}\left(\int_{\varphi}^{\theta} \eta(\theta')\, \mathrm{d}\theta'\right) \mathrm{d}\varphi\right\}. \tag{70}$$

By comparing equations (26) and (70) we notice that the Markovian model discussed in section 3 may be recovered as a particular case of our present approach which corresponds to

$$\xi(\theta' \mid \theta)\, \mathrm{d}\theta' = \begin{cases} \mathrm{d}\theta'/\theta & \text{for } \theta' \leqslant \theta \\ 0 & \text{for } \theta' > \theta \end{cases} \tag{71a}$$

and

$$v(\theta) = \theta / \langle \varphi \rangle. \tag{72}$$

By inserting equations (71*a*), (71*b*) and (72) into equation (70) we recover equation (26).

By using the expansions (29) and (30) and the integral relation (32) we can write the generating functional (70) in the standard form of a cumulant expansion,

$$Z[\eta(\theta')]=\exp\left\{\sum_{m=1}^{\infty}(-1)^m(m!)^{-1}\int_0^{\theta}\cdots\int_0^{\theta}\eta(\theta_1')\ldots\eta(\theta_m')\right.$$
$$\left.\times\left[\langle\!\langle\rho^m\rangle\!\rangle+\nu(\theta)\langle\rho^m\rangle\int_0^{\min(\theta_1',\ldots,\theta_m')}\xi(\varphi|\,\theta)\,\mathrm{d}\varphi\right]\mathrm{d}\theta_1'\ldots\mathrm{d}\theta_m'\right\} \tag{73}$$

from which we get the following expressions for the cumulants:

$$\langle\!\langle r(\theta_1')\ldots r(\theta_m')\rangle\!\rangle=\langle\!\langle\rho^m\rangle\!\rangle+\langle\rho^m\rangle\nu(\theta)\int_0^{\min(\theta_1',\ldots,\theta_m')}\xi(\varphi|\,\theta)\,\mathrm{d}\varphi. \tag{74}$$

In particular, the average shape and the dispersion are given by

$$\langle r(\theta')\rangle=\langle\!\langle\rho^m\rangle\!\rangle+\langle\rho^m\rangle\nu(\theta)\int_0^{\theta'}\xi(\varphi|\,\theta)\,\mathrm{d}\varphi \tag{75}$$

$$\langle\Delta r^2(\theta')\rangle=\langle\Delta\rho^2\rangle+[\langle\Delta\rho^2\rangle+\langle\rho\rangle^2]\nu(\theta)\int_0^{\theta'}\xi(\varphi|\,\theta)\,\mathrm{d}\varphi. \tag{76}$$

The asymptotic behaviour of equations (75) and (76) at the end of the interval $(\theta'=\theta)$ as $\theta\to\infty$ is

$$\langle r(\theta)\rangle=\langle\rho\rangle\nu(\theta)\qquad\theta'=\theta\qquad\theta\to\infty \tag{77}$$

$$\langle\Delta r^2(\theta)\rangle=\langle\rho^2\rangle\nu(\theta)\qquad\theta'=\theta\qquad\theta\to\infty \tag{78}$$

and the relative fluctuation of the shape may be approximated by

$$\langle\Delta r^2(\theta)\rangle^{1/2}/\langle r(\theta)\rangle=[\langle\rho^2\rangle^{1/2}/\langle\rho\rangle][\nu(\theta)]^{-1/2}\qquad\text{as }\theta\to\infty. \tag{79}$$

As $\nu(\theta)$ is a non-decreasing function of θ, as $\theta\to\infty$ the relative fluctuation is generally negligible and the intermittent behaviour is missing. The main new feature of the model is that for $\theta\to\infty$ the average shape is determined by the function $\nu(\theta)$, which is positive and non-decreasing but otherwise arbitrary. For instance, if $\nu(\theta)$ is an exponentially increasing function

$$\nu(\theta)=\nu(0)\exp(a\theta)\qquad a>0 \tag{80}$$

the average shape is given by a logarithmic spiral

$$\langle r(\theta)\rangle\simeq\langle\rho\rangle\nu(0)\exp(a\theta)\qquad\text{for }\theta'=\theta\qquad\theta\to\infty. \tag{81}$$

For such a logarithmic spiral the decrease of the relative shape fluctuation as $\theta\to\infty$ is exponential, i.e. much faster than in the Markovian case studied in section 3:

$$\langle\Delta r^2(\theta)\rangle^{1/2}/\langle r(\theta)\rangle=[\langle\rho^2\rangle^{1/2}/\langle\rho\rangle][\nu(0)]^{-1/2}\exp(-\tfrac{1}{2}a\theta)\qquad\text{as }\theta\to\infty. \tag{82}$$

For $\theta'<\theta$ and variable $\xi(\theta'|\,\theta)$ the description of the random shape in terms of a master equation is generally not possible. This is due to the fact that a variable probability density $\xi(\theta'|\,\theta)$ leads to a non-Markovian behaviour. However, at the end of the interval, $\theta'=\theta$, the one-point probability density is independent of $\xi(\theta'|\,\theta)$ and a master equation description is possible. By applying equations (42), (43) and (70) we get the following expression for the Laplace transform of $P_1(r, \theta)$:

$$\bar{P}_1(s, \theta) = \bar{p}(s) \exp[\nu(\theta)(p(s) - 1)]. \tag{83}$$

Equation (83) may be obtained by solving an angle-inhomogeneous master equation:

$$\frac{\partial P_1(r, \theta)}{\partial \theta} = \frac{\partial \nu(\theta)}{\partial \theta} \left[\int_0^r p(\rho) P_1(r - \rho, \theta) \, d\rho - P_1(r, \theta) \right] \tag{84}$$

with the initial condition (46).

6. Discussion

There is a partial overlapping among the three approaches presented in this paper. The first approach is amenable to a thorough analytical treatment; it has the disadvantage that the only possible average shape is a linear Archimedean spiral. The third approach circumvents this difficulty in a rather formal way by using inhomogeneous Poissonian statistics. The most complex model is the second one. Although for the second model only two different average spiral shapes are possible, it presents an interesting feature which is missing in the other two models: the absolute shape fluctuations are intermittent and the relative fluctuation does not vanish for large angles but tends towards a constant value.

In order to illustrate the different types of asymptotic behaviour we have computed the average shapes and the relative fluctuation for three different cases. Figures 1, 2 and 3 show the average asymptotic behaviour for linear Archimedean, nonlinear Archimedean and logarithmic spirals, respectively; we see that the average shapes have

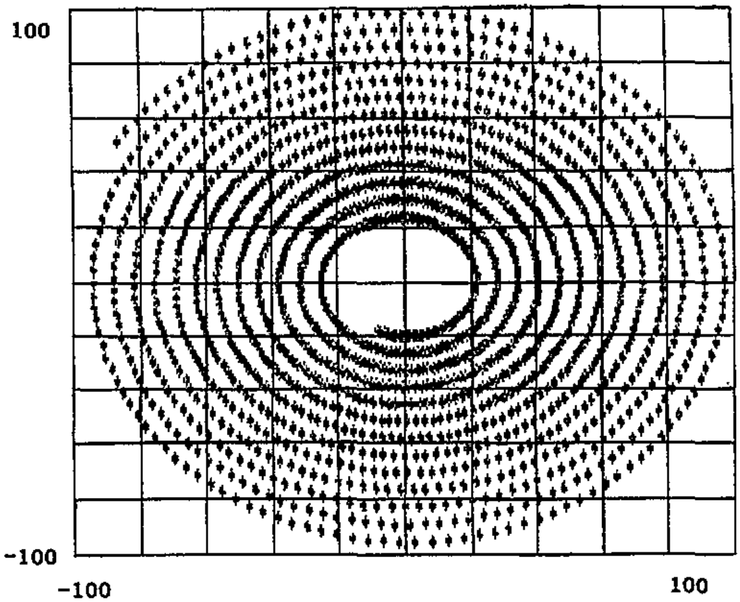

Figure 1. The average asymptotic behaviour for a linear Archimedean spiral with $\langle \rho \rangle / \langle \varphi \rangle = 1$.

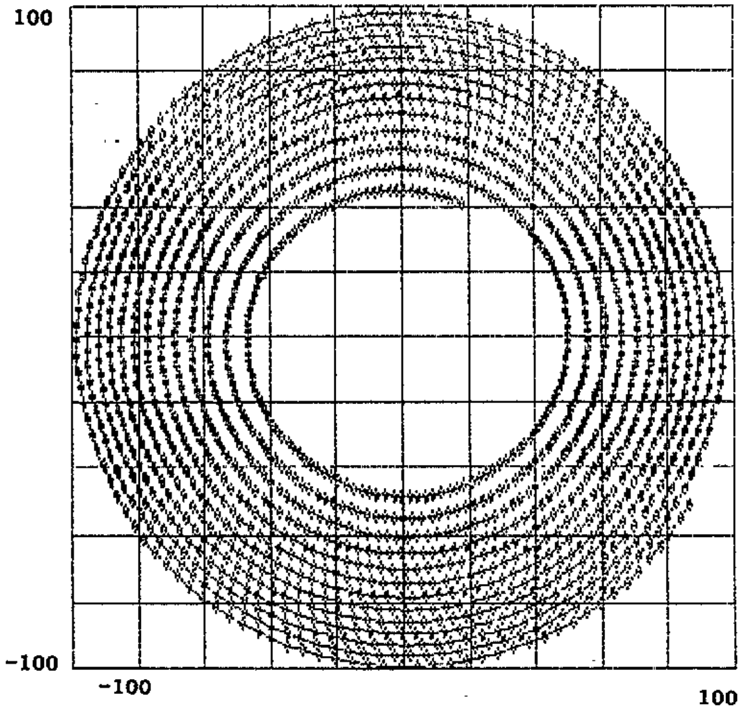

Figure 2. The average asymptotic behaviour for a nonlinear Archimedean spiral with $H=0.5$ and $\langle\rho\rangle/[B\Gamma(H+1)]=1$.

a similar behaviour in the linear and the nonlinear Archimedean cases; in contrast, the asymptotic behaviour for a logarithmic spiral is different. We notice that the analysis of the average behaviour cannot be used to identify the presence or absence of the intermittent behaviour; for that a fluctuation analysis is necessary. Figure 4 displays the asymptotic behaviour of the relative shape fluctuation in the linear Archimedean, nonlinear Archimedean and the logarithmic cases. For the nonlinear Archimedean model the fluctuations are intermittent and the relative fluctuation is asymptotically constant, whereas for the linear Archimedean and the logarithmic models the relative fluctuation decays to zero, obeying an inverse power law and an exponential law, respectively.

In this paper the analogies to the usual random walk description have been outlined, in particular the possible use of master equations. The analysis has shown that the generating functional formalism is more appropriate for describing the shape fluctuations than the master equations. The generating functional gives an integrated account of the random behaviour of the whole shape of the spiral; in contrast, the master equations give a local description.

Concerning the potential applicability of the approaches presented here, they may be used as semiphenomenological (mesoscopic) models for the different systems in which spiral shapes or motions may occur, for instance in connection with the percolation description of galactic structure (Seiden and Schulman 1990), the study of biological shapes (Thompson and D'Arcy 1992, West 1990), pattern generation (Murray 1989) and the propagation of nonlinear chemical waves (Wesfried *et al* 1988).

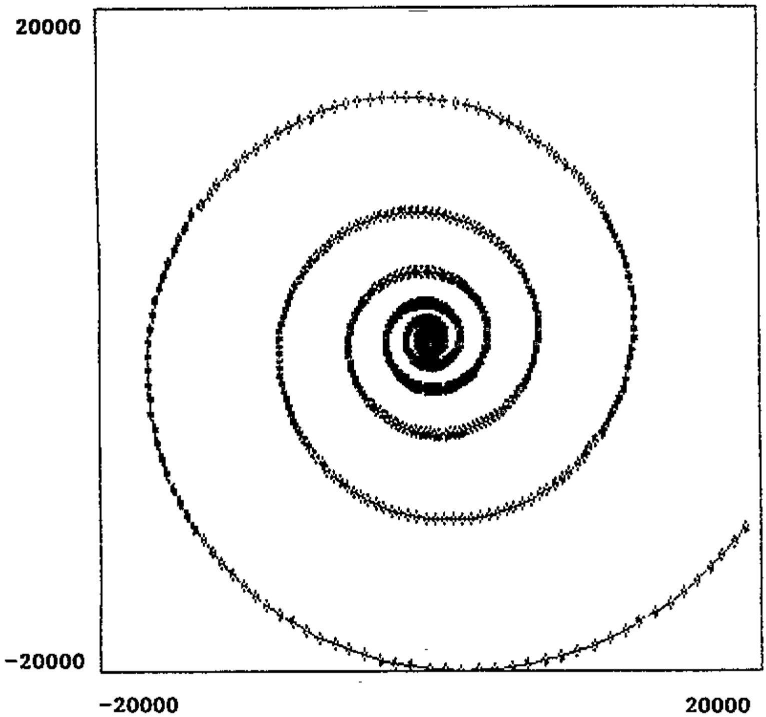

Figure 3. The average asymptotic behaviour for a logarithmic spiral with $\langle\rho\rangle\nu(0)=1$ and $a=10^{-1}$.

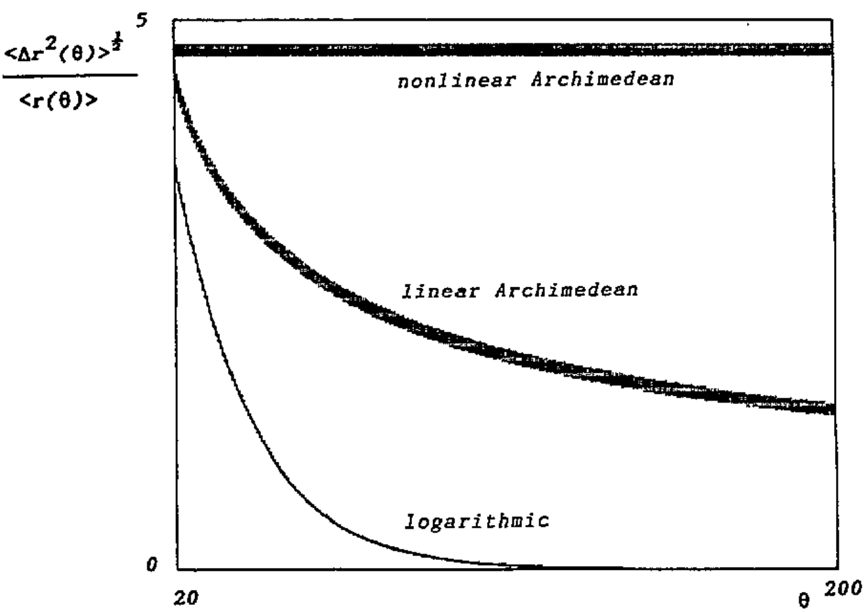

Figure 4. The asymptotic behaviour of the relative shape fluctuation for linear Archimedean, nonlinear Archimedean and logarithmic spirals with $\langle\varphi\rangle^{1/2}\langle\rho^2\rangle^{1/2}/\langle\rho\rangle=20$, $a=10^{-1}$ and $[\langle\rho^2\rangle^{1/2}/\langle\rho\rangle][\nu(0)]^{-1/2}=10$.

Acknowledgments

The author wishes to thank professors K W Kehr, J Ross and M Velarde for useful discussions, Cornelia Pascu and Stan Andrei for assistance with the graphics and computing, the Alexander von Humboldt Foundation for financial support and the Forschungszentrum Jülich for their hospitality. Thanks are also due to professor J Ross for his hospitality at Stanford University, where a previous version of this paper was written.

Appendix 1

By inserting equations (15) and (17) into equation (6) we obtain

$$
\begin{aligned}
Z[\eta(\theta)] = f(\theta) \int_0^\infty & p(r_0) \exp(-r_0 F)\, \mathrm{d}r_0 + \sum_{q=1}^{\infty} \\
& \times \int_0^\theta \ldots \int_{\theta_1}^{\theta_2} g(\theta_1) g(\theta_2-\theta_1) \ldots g(\theta_q-\theta_{q-1}) f(\theta-\theta_q) \\
& \times \int_0^\infty \ldots \int_0^{r_1} \exp[-r_0F_1 - r_1(F_2-F_1) - \ldots - r_q(F-F_q)] \\
& \times p(r_0)p(r_1-r_0)\ldots p(r_q-r_{q-1})\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_q\, \mathrm{d}r_0 \ldots \mathrm{d}r_q
\end{aligned}
\tag{A1.1}
$$

where

$$
F = \int_0^\theta \eta(\theta)\, \mathrm{d}\theta \qquad F_l = \int_0^{\theta_l} \eta(\theta)\, \mathrm{d}\theta \qquad l = 1, \ldots, q. \tag{A1.2}
$$

By using the algebraic identity

$$
\begin{aligned}
r_q(F-F_q) &+ r_{q-1}(F_q-F_{q-1}) + \ldots + r_1(F_2-F_1) + r_0F_1 \\
&= (r_q-r_{q-1})(F-F_q) + (r_{q-1}-r_{q-2})(F-F_{q-1}) \\
&\quad + \ldots + (r_1-r_0)(F-F_1) + r_0F
\end{aligned}
\tag{A1.3}
$$

we can use the integration variables

$$
\rho_0 = r_0,\ \rho_1 = r_1 - r_0,\ \ldots,\ \rho_q = r_q - r_{q-1}. \tag{A1.4}
$$

In terms of $\rho_1, \ldots, \rho_q$ the $(q+1)$-fold radial integral from equation (A1.1) factorizes into a product of $q+1$ independent integrals which can be expressed in a closed form by means of the Laplace transform of $p(r)$, $\bar{p}(s)$:

$$
\begin{aligned}
\int_0^\infty \ldots \int_0^{r_1} & \exp[-r_0F_1 - r_1(F_2-F_1) - \ldots - r_q(F-F_q)] \\
& \times p(r_0)p(r_1-r_0)\ldots p(r_q-r_{q-1})\, \mathrm{d}r_0 \ldots \mathrm{d}r_q \\
& = \prod_{l=1}^{q} [\bar{p}(F-F_l)]\bar{p}(F).
\end{aligned}
\tag{A1.5}
$$

By inserting equation (A1.5) into equation (A1.1) and expressing the factors $F, F_1, \ldots, F_q$ in terms of $\eta(\theta)$ by means of equation (A1.2) we get equation (18).

Appendix 2

Equation (32) may be proved by mathematical induction. For $m=2$ the integral I_m becomes

$$I_2=\int_0^\theta \mathrm{d}\varphi \int_\varphi^\theta \mathrm{d}\theta_1 \int_\varphi^\theta \mathrm{d}\theta_2 \ldots = \iiint_{D_2} \ldots \mathrm{d}\varphi\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \tag{A2.1}$$

where the integration domain D_2 is given by

$$D_2: \theta \geqslant \theta_1 \geqslant \varphi \qquad \theta \geqslant \theta_2 \geqslant \varphi \qquad \theta \geqslant \varphi \geqslant 0. \tag{A2.2}$$

By changing the order of integration in equation (A2.1) we obtain

$$\iiint_{D_2} \ldots \mathrm{d}\varphi\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2$$

$$= \int_0^\theta \mathrm{d}\theta_1 \int_0^\theta \mathrm{d}\theta_2 \left[h(\theta_1-\theta_2) \int_0^{\theta_2} \mathrm{d}\varphi \ldots + h(\theta_2-\theta_1) \int_0^{\theta_1} \mathrm{d}\varphi \ldots \right] \tag{A2.3a}$$

where $h(x)$ is the usual Heaviside function. We notice that the sum of the two integrals over φ in the RHS of equation (A2.3a) may be written as

$$h(\theta_1-\theta_2) \int_0^{\theta_2} \mathrm{d}\varphi \ldots + h(\theta_2-\theta_1) \int_0^{\theta_1} \mathrm{d}\varphi \ldots = \int_0^{\min(\theta_1,\theta_2)} \mathrm{d}\varphi \ldots \tag{A2.3b}$$

that is, equation (32) is true for $m=2$.

Now we should prove that the validity of equation (32) for a given m implies the validity of equation (32) for $m \to m+1$. I_{m+1} is given by

$$I_{m+1} = \int_0^\theta \mathrm{d}\varphi \int_\varphi^\theta \mathrm{d}\theta_1 \ldots \int_\varphi^\theta \mathrm{d}\theta_{m+1} \ldots. \tag{A2.4}$$

As equation (32) is assumed to be valid for a given m, it can be used for changing the order of the first $m+1$ integrals over $\varphi, \theta_1, \ldots, \theta_m$ in equation (A2.4). We get

$$I_{m+1} = \int_0^\theta \mathrm{d}\theta_1 \ldots \int_0^\theta \mathrm{d}\theta_m \int_0^{\theta_m^*} \mathrm{d}\varphi \int_\varphi^\theta \mathrm{d}\theta_{m+1} \ldots \tag{A2.5}$$

with

$$\theta_m^* = \min(\theta_1, \ldots, \theta_m). \tag{A2.6}$$

By changing the integration order over φ and θ_{m+1} in equation (A2.5) we have

$$I_{m+1}=\int_0^\theta \mathrm{d}\theta_1 \ldots \int_0^\theta \mathrm{d}\theta_{m+1}\left[h(\theta_m^*-\theta_{m+1})\int_0^{\theta_{m+1}} \mathrm{d}\varphi \ldots + h(\theta_{m+1}-\theta_m^*)\int_0^{\theta_m^*} \mathrm{d}\varphi \ldots\right]$$

$$=\int_0^\theta \mathrm{d}\theta_1 \ldots \int_0^\theta \mathrm{d}\theta_{m+1}\int_0^{\theta_{m+1}^*} \mathrm{d}\varphi \ldots \tag{A2.7}$$

where

$$\theta_{m+1}^*=\min(\theta_m^*, \theta_{m+1})=\min(\theta_1, \ldots, \theta_{m+1}). \tag{A2.8}$$

It follows that the validity of equation (32) for a given m implies the validity of the same equation for $m+1$. As equation (32) is true for $m=2$, it turns out that, according to the method of complete induction, equation (32) holds true for any positive integer m.

Appendix 3

By differentiating equation (16) and expressing in the resulting equation the function $g(\varphi)$ from equation (61), we get a relationship between $f(\varphi)$ and $\lambda(\varphi)$:

$$\mathrm{d}f(\varphi)/\mathrm{d}\varphi=-\lambda(\varphi)f(\varphi). \tag{A3.1}$$

By integrating equation (A3.1) and taking into account that $f(0)$ should equal 1, we obtain

$$f(\varphi)=\exp\left[-\int_0^\varphi \lambda(\varphi)\,\mathrm{d}\varphi\right]. \tag{A3.2}$$

By combining equations (61) and (A3.2) we can express $g(\varphi)$ in terms of $\lambda(\varphi)$:

$$g(\varphi)=\lambda(\varphi)\exp\left[-\int_0^\varphi \lambda(\varphi)\,\mathrm{d}\varphi\right]. \tag{A3.3}$$

Now we integrate equation (63) with the initial condition (65) and use the expression (A3.1) for $f(\varphi)$. We have

$$\mathscr{P}_1(r, \varphi, \theta)=h(\theta-\varphi)C(r, \theta-\varphi)f(\varphi)+h(\varphi-\theta)p(r)\delta(\varphi-\theta)f(\varphi) \tag{A3.4}$$

where

$$C(r, \theta)=\mathscr{P}_1(r, 0, \theta). \tag{A3.5}$$

By inserting equation (A3.4) into equations (66) and (64) we get an expression for $P_1(r, \theta)$ in terms of $C(r, \theta)$ and an integral equation for $C(r, \theta)$, respectively:

$$P_1(r, \theta)=\int_0^\theta f(\varphi)C(r, \theta-\varphi)\,\mathrm{d}\varphi+f(\theta)p(r) \tag{A3.6}$$

$$C(r, \theta)=\int_0^r\int_0^\theta g(\varphi)p(\rho)C(r-\rho, \theta-\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\rho+g(\theta)\int_0^r p(\rho)p(r-\rho)\,\mathrm{d}\rho. \tag{A3.7}$$

By means of a double Laplace transformation, equations (A3.6) and (A3.7) are simplified:

$$\bar{\bar{P}}_1(s, x) = \bar{f}(x)\bar{\bar{C}}(s, x) + \bar{f}(x)\bar{p}(s) \tag{A3.8}$$

$$\bar{\bar{C}}(s, x) = \bar{g}(x)\bar{p}(s)\bar{\bar{C}}(s, x) + \bar{g}(x)\bar{p}^2(s). \tag{A3.9}$$

By solving equation (A3.9) for $\bar{\bar{C}}(s, x)$ and inserting the resulting expression into equation (A3.8) we recover equation (60).

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